From guarded to well-founded

Formalizing CoQ’s guard condition

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1. Guard condition and trust

2. Translating guarded definitions

3. Current state of the implementation
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In Coq, we have Fixpoint and match instead of recursors.

```coq
Fixpoint plus (n m : nat) {struct n} : nat :=
match n with
| O    ⇒ m
| S n'  ⇒ S (plus n' m)
end.
```

The two presentations should be equivalent.

```coq
Fixpoint nat_rect (P : nat → Type) (fO : P O) (fS : ∀ n, P n → P (S n)) (n : nat) : P n :=
match n with
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| S n'  ⇒ fS n' (nat_rect P fO fS n')
end.
```
Recursive definitions

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match n with
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end.
```
Deep recursion

Inductive Even : nat → Prop :=
zeven : Even 0 | seven \{n : nat\} : Even n → Even (S (S n)).

Inductive Odd : nat → Prop :=
oneodd : (Odd (S O)) | sodd \{n : nat\} : Odd n → Odd (S (S n)).

Definition EO (n : nat) := \{Even n\} + \{Odd n\}.

Definition aux \{n : nat\} (H : EO n) : EO (S (S n)) :=
match H with left p ⇒ left (seven p) | right p ⇒ right (sodd p) end.

Fixpoint evod (n : nat) : EO n :=
match n with
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  match m with
  | O ⇒ right oneodd
  | S p ⇒ aux (evod p)
end end.
The guard condition

We need to make sure that every Fixpoint terminates.

- Syntactic condition on the body of a Fixpoint.
- For each variable in the current context, track whether it is a subterm of the recursive argument.
- For each recursive call, check that the recursive argument is a subterm.
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\end{itemize}

\texttt{Fixpoint \ plus (n m : nat) \{struct n\} : nat :=}
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match n with
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end.
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  - A match with certain restrictions can be a subterm if all branches are a subterm.
- ...?

⇒ We want a current justification of the guard condition.
Our proposal: relying on well-foundedness

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Use the subterm relation as a well-founded relation. For each such subterm, add a \texttt{CoQ} proof that it is a subterm.

\begin{verbatim}
Definition nat_subterm : relation nat.
Theorem wf_nat_subterm : well_founded nat_subterm.

(* Large subterms. *)
Definition nat_subterm_eq : relation nat.

Definition nat_case (N : nat) (x : nat) (Hsub : nat_subterm_eq x N) (P : nat \to Type) (f0 : P 0) (fS : forall (y : nat), nat_subterm y N \to P (S y)) : P x.

Definition plus_body (n m : nat) (F : forall (n' m' : nat), nat_subterm n' n \to nat) : nat.
\end{verbatim}
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Seen from the top, rather straightforward:

- Recursively go through the function body, while collecting subterm information in the context.
- At a recursive call, check that we have a subterm in the recursive position.
Back to the guard condition

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To check the guard on...

```plaintext
match c return p with
| C_i x y z ⇒ t
end
```

...check that c and p are guarded, then compute the recursive information on x, y, z depending on the information on c, and check t under this context.
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\[(\text{fun } x \Rightarrow t)\]

...check that \(t\) is guarded under a context where \(x\) is not a subterm.
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To check the guard on...

```haskell
match c with
| C f ⇒ (fun x ⇒ t)
end y
```

...check that `t` is guarded under a context where `x` is not a subterm...unless there was some term applied to a `match` previously.
Back to the guard condition

Seen from the top, rather straightforward:

- Recursively go through the function body, while collecting subterm information in the context.
- At a recursive call, check that we have a subterm in the recursive position.

To check the guard on...

\[ f \, x \, y \, z \]

...when \( f \) is the function being defined, check that the argument in recursive position is a subterm.
The subterm relation

For each inductive type, we can systematically define a direct subterm relation.

\begin{verbatim}
Inductive nat : Set :=
0 : nat | S : nat -> nat.

Inductive nat_direct_subterm : relation nat :=
nat_ds_1 : forall (n : nat), nat_direct_subterm n (S n).
\end{verbatim}

\footnote{See for instance \textit{Equations}}
The subterm relation

For each inductive type, we can systematically define a direct subterm relation.

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```

The subterm relation is its transitive closure.

```lean
Definition nat_subterm : relation nat :=
  clos_trans nat nat_direct_subterm.
```

This is already done automatically by some tools\(^1\), as well as proving its well-foundedness.

\(^1\)See for instance \textsc{Equations}
Fixpoint evod (n : nat) : EO n :=
match n with
| O ⇒ left zeven
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  match m with
  | O ⇒ right oneodd
  | S p ⇒ aux (evod p)
end end.

Definition evod_body (n : nat)
(F : forall (m : nat), nat_subterm m n → EO m) : EO n :=
nat_case n n r_refl EO
  (left zeven)
  (fun m Hsub ⇒ nat_case n m (r_step Hsub) (fun m ⇒ EO (S m))
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Fixpoint evod (\(n : \text{nat}\)) : EO n :=
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Translation

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What about mutual inductive types?

We have two choices:

- Define heterogeneous subterm relations for each pair of mutually inductive types.
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  - need to introduce a more complicated \texttt{Fix} to handle several types at once

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  - a bit more verbose and complicated
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In any case, there are some problems to solve with sorts.
What about nested inductive types?

```latex
Inductive rose : Set :=
| node : list rose → rose.
```
What about nested inductive types?

\textbf{Inductive} \texttt{rose} : \textit{Set} :=
\begin{itemize}
    \item \texttt{node} : \textit{list rose} \rightarrow \texttt{rose}.
\end{itemize}

► The correct solution is not clear yet...
What about nested inductive types?

```latex
Inductive rose : Set :=
| node : list rose → rose.
```

- The correct solution is not clear yet...
- For now, we inline the definition of the nested type.

```latex
Inductive Rose' : Set :=
| node' : listRose' → Rose'
with listRose' : Set :=
| nil' : listRose'
| cons' : Rose' → listRose' → listRose'.
```

- Ideally, we would like to reuse generically what was built for lists.
Inductive True2 : Prop :=
I2: (False → True2) → True2.

(* Using prop_ext : forall P Q, (P ↔ Q) → P = Q. *)
Theorem Heq: (False → True2) = True2.
Evolution of the guard condition

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(* Using prop_ext : forall P Q, (P ↔ Q) → P = Q. *)
Theorem Heq: (False → True2) = True2.

Fixpoint con (x : True2) : False :=
  match x with
  I2 f ⇒ con (match Heq in _=T return T with eq_refl ⇒ f end)
end.
Outline

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Template Coq

- Quoting library for \texttt{Coq}.
- Allows the user to manipulate quotations and the global environment.
- Will include a certified checker for \texttt{Coq}.

Assuming completion of this work...

You only need to trust the well-foundedness of \texttt{Acc} to trust any recursive definition accepted by Template Coq.
Regular trees

- Used to represent the recursive structure of a term.
- Infinite trees with finitely many subtrees.
- Represented in Coq’s kernel by a finite datastructure with de Bruijn indices.

```coq
Inductive rtree (X : Set) : Set :=
| rNode (x : X) (sons : list (rtree X)) : rtree X
| rParam (i : nat) (j : nat) : rtree X
| rRec (j : nat) (defs : list (rtree X)) : rtree X.
```

From a client viewpoint, only see it as a tree of \texttt{rNode} with some payload of type \texttt{X}.
An example of regular tree

\[
\text{Inductive } \text{foo : Set :=}
\]
\[
| \ A \ : \ \text{foo} \rightarrow \text{bar} \rightarrow \text{foo}
\]
\[
| \ B \ : \ \text{foo}
\]
\[
\text{with bar : Set :=}
\]
\[
| \ C \ : \ \text{nat} \rightarrow \text{foo} \rightarrow \text{bar}.
\]
A closed tree is either:

- some rNode $x$ sons: the payload is $x$ and its children are sons
- some rRec $j$ defs: in that case we can consider the unfolding of this recursive node, that is $\text{defs}(j)$ where each $\text{defs}(i)$ is substituted for rParam 0 i.

This unfolding operation guarantees that we can always provide an actual rNode
A closed tree is either:

- some \texttt{rNode x sons} : the payload is \texttt{x} and its children are \texttt{sons}
- some \texttt{rRec j defs} : in that case we can consider the unfolding of this recursive node, that is \texttt{defs.(j)} where each \texttt{defs.(i)} is substituted for \texttt{rParam 0 i}.

This unfolding operation guarantees that we can always provide an actual \texttt{rNode} if the tree verifies some acyclicity condition.
The next steps are:

- Implement the positivity criterion.
  - Not too difficult, except that it relies itself on the strong normalization of Coq terms.

- Implement the guard condition.
- Implement the translation of guarded functions.
- Of course, prove all this correct all along.
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Of course, prove all this correct all along.